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Penalized complementarity functions on symmetric cones

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Abstract We show that penalized functions of the Fischer–Burmeister and the natural residual functions defined on symmetric cones are complementarity functions. Boundedness of the solution set of a symmetric cone complementarity problem, based on the penalized natural residual function, is proved under monotonicity and strict feasibility. The proof relies on a trace inequality on Euclidean Jordan algebras.

Keywords Complementarity problem · Complementarity functions · Merit functions · Symmetric cones

Mathematics Subject Classification (2000) Primary: 90C33

1 Introduction

Since the pioneering works by Faybusovich [7–9] which adapted Jordan algebra (or symmetric cone) theory to describe interior point method, a new trend has started in convex optimization. The formalism of Jordan algebra is getting more and more useful to successfully extend some interior-point algorithms for LP to quadratic and semidefinite programming, as pointed out in [1]. Baes [1] explained briefly a justification for using this more general framework of Jordan algebra. Moreover, in a recent work, Sun and Sun [16] investigated differentiability and semismoothness of Löwner's operator and spectral functions, and generalized some of their previous results [15] under the framework of Euclidean Jordan algebra [12]. They

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showed that many optimization-related classical results in the symmetric matrix space can be extended into symmetric cone settings. They also made a comment that more research is necessary for the analysis of optimization problems within the unifying framework of Euclidean Jordan algebra.

Motivated by the above mentioned works, we are trying to deal with the symmetric cone complementarity problem (for short, SCCP) which contains NCP, SDCP (semidefinite complementarity problem) and SOCCP (second-order cone complementarity problem) as its typical special cases. We focus our attention on the so-called 'complementarity function' (C-function). The terminology *C-function* was formally coined in the book of Facchinei and Pang [5]. This monograph [5] provides the most comprehensive treatment of complementarity problems on a finite-dimensional Euclidean space. In this paper, we show that the penalized versions of the natural residual function for SOCCP [4] and the Fischer–Burmeister NCP-function [3] can be extended to SCCP in a Euclidean Jordan algebra. Of course, there exist several penalized C-functions except the two treated in this paper. However, the two were extensively analyzed in the NCP setting. So we restrict ourselves to the penalized Fischer–Burmeister function and the penalized natural residual function in the present work. Moreover, under monotonicity and strict feasibility conditions, we derive the boundedness of level sets of the natural merit function induced by the penalized natural residual function from a trace inequality in Euclidean Jordan algebras. As a direct consequence, the boundedness of solution sets of a monotone SCCP under strict feasibility is obtained. Readers may refer to [5] for the case of NCP. As is well-known, the boundedness of level sets of a merit function is an important property since it ensures that the sequence generated by a possible descent method has at least one accumulation point.

Our approach sheds light on the close relationship between the boundedness of level sets of merit functions of SCCP and basic trace inequalities. Proving methods are purely Jordan-algebraic according to the motivation of this work. Throughtout this paper, we depend heavily upon so-called "trace argument" which features our work. This argument is simple but very useful.

2 Euclidean Jordan algebras

We recall certain basic notions and well-known facts concerning Jordan algebras from the book by Faraut and Korányi [6]. A *Jordan algebra* V with an identity element e over the field $\mathbb{R}$ or $\mathbb{C}$ is a commutative algebra satisfying $x^2(xy) = x(x^2 y)$ for all $x, y \in V$. Every Jordan algebra is power associative which means that the algebra generated by x and e is associative. Denote $L(x)$ by the multiplication operator $L(x)y = xy$, and set $P(x) = 2L(x)^2 - L(x^2)$ for $x \in V$. An element $x \in V$ is said to be *invertible* if there exists an element y in the subalgebra generated by x and e such that $xy = e$.

A finite-dimensional real Jordan algebra V is called a *Euclidean Jordan algebra* if it carries an associative inner product $\langle \cdot, \cdot \rangle$ on V, namely $\langle xy, z \rangle = \langle y, xz \rangle$ for all $x, y, z \in V$. An element $c \in V$ is *idempotent* if $c^2 = c$, and two idempotents c and c' are *orthogonal* if $cc' = 0$. If an idempotent c cannot be written by a sum of two non-zero idempotents then c is called *primitive*. One says that $c_1, \ldots, c_k$ is a *complete system of orthogonal idempotents* if $e = \sum_{i=1}^{k} c_i$, $c_i c_j = \delta_{ij} c_i$. A *Jordan frame* is a complete system of orthogonal primitive idempotents. The following two theorems are fundamental in the theory of Euclidean Jordan algebra. Actually, we introduce more detailed statements in [2] rather than the original ones in [6] as follows:

Theorem 2.1 (Spectral theorem, first version [6, Theorem III.1.1]) *For an element x of a Euclidean Jordan algebra V there exist unique real numbers $\lambda_1 > \cdots > \lambda_k$ and a unique complete system of orthogonal idempotents $c_1, \ldots, c_k$ such that $x = \sum_{i=1}^{k} \lambda_i c_i$. The uniqueness is in the following sense: if there exist a complete system of orthogonal idempotents $\{e_1, \ldots, e_s\}$ and distinct real numbers $\eta_1 > \cdots > \eta_s$ such that $x = \sum_{i=1}^{s} \eta_i e_i$, then $k = s$ and $\eta_i = \lambda_i$ and $e_i = c_i$ for all $1 \le i \le k$.*

Theorem 2.2 (Spectral theorem, second version [6, Theorem III.1.2]) *For an element x of a Euclidean Jordan algebra V there exist a Jordan frame $c_1, \ldots, c_r$ (r is fixed and called the rank of V) and real numbers $\lambda_1 \ge \cdots \ge \lambda_r$ such that $x = \sum_{i=1}^{r} \lambda_i c_i$. If there exist a Jordan frame $e_1, \ldots, e_r$ and real numbers $\eta_1 \ge \cdots \ge \eta_r$ such that $x = \sum_{i=1}^{r} \eta_i e_i$, then $\eta_i = \lambda_i$ for all i and $\sum_{\{j|\eta_j=\alpha\}} e_j = \sum_{\{j|\eta_j=\alpha\}} c_j$ for each real number α.*

Let $\operatorname{tr}(x) = \sum_{i=1}^{r} \lambda_i$, the trace of $x = \sum_{i=1}^{r} \lambda_i c_i$ in the second spectral theorem. The trace inner product $\operatorname{tr}(xy)$ is associative and in this case every Jordan automorphism is an orthogonal transformation with respect to the trace inner product. *Throughout this paper we assume that V is a Euclidean Jordan algebra equipped with the trace inner product $\langle \cdot, \cdot \rangle$.*

Let Ω be the open convex cone of invertible squares of a Euclidean Jordan algebra V. Then Ω is a symmetric cone, that is, the group $G(\Omega) := \{g \in \mathrm{GL}(V) : g(\Omega) = \Omega\}$ acts transitively on it and Ω is a self-dual cone with respect to the trace inner product.

Proposition 2.3 ([6, Proposition III.2.2]) *For $x \in V$, $x \in \overline{\Omega}$ (resp. Ω) if and only if $L(x)$ is positive semidefinite (resp. definite).*

For any function $\hat{f} : I \to J$ defined on intervals of $\mathbb{R}$, we define a function on all $x \in V$ with spectrum contained in I by $f(x) = \sum_{i=1}^{r} \hat{f}(\lambda_i) c_i$ where $x = \sum_{i=1}^{r} \lambda_i c_i$ the spectral decomposition of second type. This function is called *Löwner's operator* in [16]. If $\hat{f}$ is bijective then f is a bijection from all elements with spectrum contained in I to all elements with spectrum contained in J. For example, $x \to x^p$, $p > 0$ on nonnegative reals expend bijections on the set of all elements with nonnegative spectrum.

Recall the Löwner partial order on V defined by $x \preceq y :\Longleftrightarrow y - x \in \overline{\Omega}$, and $x \prec y :\Longleftrightarrow y - x \in \Omega$. It is not difficult to see that $x \in \Omega$ (resp. $x \in \overline{\Omega}$) if and only if there exists a Jordan frame $\{c_i\}$ and $\lambda_i > 0$ (resp. $\lambda_i \ge 0$) for $i = 1, \ldots, r$ such that $x = \sum_{i=1}^{r} \lambda_i c_i$.

Lemma 2.4 *Let p be a positive real number.*

(i) Each element $x \succeq 0$ has a unique p-th root denoted by $x^{1/p}$ in $\overline{\Omega}$. If $x \in \overline{\Omega}$ has a spectral decomposition $x = \sum_{i=1}^{r} \lambda_i c_i$, then $x^{1/p} = \sum_{i=1}^{r} \lambda_i^{1/p} c_i$.
(ii) (The Löwner-Heinz inequality, [13])

$$0 \preceq x \preceq y \Longrightarrow x^p \preceq y^p, \quad 0 \le p \le 1.$$

For $x \in V$, we denote $|x|$ by $|x| = (x^2)^{1/2}$ and

$$x_+ = \frac{x + |x|}{2}, \quad x_- = \frac{|x| - x}{2}. \tag{2.1}$$

If x has a spectral decomposition $x = \sum_{i=1}^{r} \lambda_i c_i$ then

$$x_+ = \sum_{i=1}^{r} (\lambda_i)_+ c_i, \quad x_- = \sum_{i=1}^{r} (\lambda_i)_- c_i, \quad |x| = \sum_{i=1}^{r} |\lambda_i| c_i \tag{2.2}$$

where for any scalar λ, $\lambda_+ = \max\{0, \lambda\}$, $\lambda_- = \max\{0, -\lambda\}$. Since $x = x_+ - x_-$ and $\langle x_+, x_- \rangle = 0$, by the Moreau decomposition, x_+ and $-x_-$ are the projections of x onto $\overline{\Omega}$ and $-\overline{\Omega}$, respectively. Moreover, $x_+ x_- = 0$.

3 Complementarity functions on symmetric cones

According to the definition of *complementarity problem* in [5, 1.1.2 Definition], the *symmetric cone complementarity problem* (for short, SCCP) on a symmetric cone Ω is defined to be the problem finding vectors $x, y \in V$ such that

$$x \in \overline{\Omega}, \quad y \in \overline{\Omega}, \quad \langle x,\ y\rangle = 0, \quad y = F(x) \tag{3.3}$$

where $F : V \to V$ is a continuously differentiable mapping. This is equivalent to the form:

$$\text{Find } x \succeq 0, \quad \text{such that} \quad F(x) \succeq 0 \quad \text{and} \quad \langle x,\ F(x)\rangle = 0. \tag{3.4}$$

A pair of elements (x, y) satisfies the *complementarity condition* (CC) if it satisfies the defining condition of the complementarity problem Eq. (3.3):

$$x \in \overline{\Omega}, \quad y \in \overline{\Omega}, \quad \langle x,\ y\rangle = 0. \tag{3.5}$$

Proposition 3.1 *The followings are equivalent:*

(i) $x, y \succeq 0$ *and* $\langle x,\ y\rangle = 0$.
(ii) $x, y \succeq 0$ *and* $xy = 0$.
(iii) $x + y = (x^2 + y^2)^{1/2}$.
(iv) $x = [x - y]_+$.
(v) $x + y \succeq 0$, *and* $xy = 0$.
(vi) $x, y \succeq 0$ *and* $[L(x), L(y)] = L(x)L(y) - L(y)L(x) = 0$.
(vii) $x, y \succeq 0$ *and* $x^t y^s = 0$ *for all nonnegative real numbers* s, t.

Proof The equivalences from (i) to (vi) appear in [11, Proposition 6]. Suppose that $x, y \succeq 0$ and $xy = 0$. Let $A := \{t \geq 0 \mid x^t y = 0\}$. By continuity of the Jordan product, it is a closed subset of $[0, \infty)$. We claim that A contains all positive dyadic powers $\{\frac{m}{2^n} | n, m \in \mathbb{N}\}$. From

$$\begin{aligned} 0 &= \langle x,\ y\rangle = \langle x,\ (y^{1/2})^2\rangle = \langle L(y^{1/2})x,\ y^{1/2}\rangle = \langle xy^{1/2},\ y^{1/2}\rangle \\ &= \langle L(x)y^{1/2},\ y^{1/2}\rangle = \langle L(x)^{1/2}y^{1/2},\ L(x)^{1/2}y^{1/2}\rangle \end{aligned}$$

where we used the fact that $L(x)$ is positive semidefinite and hence it has the square root $L(x)^{1/2}$, we have $L(x)^{1/2}y^{1/2} = 0$ or $L(x)y^{1/2} = xy^{1/2} = 0$. Similarly, $x^{1/2}y = 0$. By induction, $x^{\frac{1}{2^n}} y = 0$ for all positive integers n. For positive integer $m \geq 2$, $0 = \langle xy,\ x^{m-1}\rangle = \langle y,\ x^m\rangle$ implies $x^m y = 0$. Therefore $x^{\frac{m}{2^n}} y = 0$ for all positive integers n and m. By the density of A in the space of non-negative real numbers, we conclude that $x^t y = 0$ for all non-negative real number t. □

A function $\phi : V \times V \to V$ is called a complementarity function (*C-function*) (see [5, 1.5.1 Definition]) if

$$\phi(x, y) = 0 \ \text{ if and only if } \langle x,\ y\rangle = 0,\ x \succeq 0,\ y \succeq 0. \tag{3.6}$$

It is immediate from Proposition 3.1 that the functions defined by

$$\begin{aligned} \phi(x, y) &= x + y - (x^2 + y^2)^{1/2}, \\ \phi(x, y) &= x - [x - y]_+ \end{aligned}$$

are C-functions, called the Fischer–Burmeister function and the natural residual function, respectively. In relation to the Fischer–Burmeister function, we also have the following general observation:

Theorem 3.2 *For positive integer $n > 1$, we define $\psi_n(x, y) = x + y - (|x|^n + |y|^n)^{1/n}$. Let $x, y \in V$.*

(1) If $x, y \succeq 0$ and $xy = 0$, then $\psi_n(x, y) = 0$.
(2) If $\psi_n(x, y) = 0$, then $x, y \succeq 0$ and $(x + y)^n = x^n + y^n$.

Furthermore, $\psi_n(x, y) = x + y - (|x|^n + |y|^n)^{1/n}$ is a C-function if $n \leq 4$.

Proof (1) Suppose that $x, y \succeq 0$ and $xy = 0$. Observe that $x = |x|$ and $y = |y|$. For $i \geq 1$, we have, by means of Proposition 3.1, $x^i y = 0$ and $y^i x = 0$ for $i = 1, 2, \ldots$. This implies by induction that $(x + y)^n = x^n + y^n$. Indeed,

$$\begin{aligned}(x + y)^n &= (x + y)(x + y)^{n-1} = (x + y)(x^{n-1} + y^{n-1}) \\ &= x^n + y^n + xy^{n-1} + yx^{n-1} = x^n + y^n.\end{aligned}$$

Therefore $x + y = (x^n + y^n)^{1/n} = (|x|^n + |y|^n)^{1/n}$. That is, $\psi_n(x, y) = 0$.

(2) Suppose that $x + y = (|x|^n + |y|^n)^{1/n}$. Setting $w = (|x|^n + |y|^n)^{1/n}$, we have $w^n = |x|^n + |y|^n \succeq |x|^n$ and $w^n = |x|^n + |y|^n \succeq |y|^n$. By the Löwner-Heinz inequality, $w \succeq |x|$ and $w \succeq |y|$. Since $|x| \succeq x$ and $|y| \succeq y$, we then have

$$x = w - y \succeq w - |y| \succeq 0, \quad y = w - x \succeq w - |x| \succeq 0.$$

Thus $|x| = x$ and $|y| = y$, so $(x + y)^n = |x|^n + |y|^n = x^n + y^n$.

Next, suppose that $\psi_n(x, y) = 0$ for $n = 2, 3, 4$. By (2), $x, y \succeq 0$ and $(x+y)^n = x^n + y^n$.

Case 1 $n = 2$. By Proposition 3.1, ψ_2 is a C-function.

Case 2 $n = 3$. From $(x + y)^3 = x^3 + y^3$, we have $x^2 y + 2(xy)x + 2(xy)y + y^2 x = 0$.

The associative property of the inner product yields

$$\begin{aligned}0 &= \langle x^2 y + 2(xy)x + 2(xy)y + y^2 x,\ e\rangle \\ &= \langle x^2,\ y\rangle + 2\langle xy,\ x\rangle + 2\langle xy,\ y\rangle + \langle y^2,\ x\rangle \\ &= \langle x^2,\ y\rangle + 2\langle y,\ x^2\rangle + 2\langle x,\ y^2\rangle + \langle y^2,\ x\rangle \\ &= 3\langle x^2,\ y\rangle + 3\langle x,\ y^2\rangle.\end{aligned}$$

Since $x, y \succeq 0$, the terms $\langle x^2,\ y\rangle$ and $\langle x,\ y^2\rangle$ are non-negative and hence must be zero. It follows from Proposition 3.1 together with its proof that $x^2 y = 0$, hence $xy = 0$.

Case 3 $n = 4.(x + y)^4 = x^4 + y^4$ implies that $2x^2 y^2 + 4(xy)^2 + 4(xy)x^2 + 4(xy)y^2 = 0$.

So,

$$\begin{aligned}0 &= \langle 2x^2 y^2 + 4(xy)^2 + 4(xy)x^2 + 4(xy)y^2,\ e\rangle \\ &= 2\langle x^2,\ y^2\rangle + 4\langle xy,\ xy\rangle + 4\langle xy,\ x^2\rangle + 4\langle xy,\ y^2\rangle \\ &= 2\langle x^2,\ y^2\rangle + 4\langle xy,\ xy\rangle + 4\langle y,\ x^3\rangle + 4\langle x,\ y^3\rangle\end{aligned}$$

Since all terms are non-negative, $xy = 0$. □

Based upon Theorem 3.2, we raise the following question.

Question. Is the function ψ_n a C-function for any positive integer $n \geq 2$?

Now we turn our attention to the penalized versions of the previous two C-functions (the Fischer–Burmeister function and the natural residual function) which are the main concerns of this paper. The penalized versions were originally considered in [3] and [4] concerned with NCP and SOCCP, respectively. We show that the penalized versions are still C-functions on symmetric cones.

Proposition 3.3 *Let $\lambda \in (0, 1)$. Define penalized version ϕ_λ of the natural residual function by $\phi_\lambda(x, y) = \lambda(x - [x - y]_+) + (1 - \lambda)x_+y_+$. For any $x, y \in V$, we have*

$$||\phi_\lambda(x, y)|| \geq \lambda \max\{||x_-||, ||y_-||\}.$$

Proof Similar to the proof of Proposition 2.4 of [4]. □

Theorem 3.4 *The function ϕ_λ is a C-function.*

Proof Suppose that (x, y) satisfies (CC). Then $x_+y_+ = xy = 0$ and hence $\phi_\lambda(x, y) = \lambda(x - [x - y]_+) = 0$ holds from Proposition 3.1 (iv).
Assume that $\phi_\lambda(x, y) = \lambda(x - [x - y]_+) + (1 - \lambda)x_+y_+ = 0$. By Proposition 3.3, $x \geq 0$ and $y \geq 0$. Thus $\lambda[x - y]_+ = \lambda x + (1 - \lambda)xy$ or

$$[x - y]_+ = x + \alpha xy, \quad \alpha = \frac{1 - \lambda}{\lambda}. \tag{3.7}$$

From $[x - y]_+ = \frac{1}{2}(x - y + ((x - y)^2)^{1/2})$, we have $((x - y)^2)^{1/2} = x + 2\alpha xy + y$. Thus

$$\begin{aligned} x^2 + y^2 - 2xy = (x - y)^2 &= (x + 2\alpha xy + y)^2 \\ &= x^2 + y^2 + 4\alpha^2(xy)^2 + 4\alpha x(xy) + 2xy + 4\alpha y(xy). \end{aligned}$$

This implies that

$$\begin{aligned} 0 &= \alpha(xy)^2 + x(xy) + y(xy) + \alpha^{-1}xy = [\alpha xy + x + y + \alpha^{-1}e](xy) \\ &= L(\alpha xy + x + y + \alpha^{-1}e)(xy), \end{aligned}$$

Observe that the element $\alpha xy + x + y + \alpha^{-1}e \in \Omega$ because $[x - y]_+ = x + \alpha xy \in \overline{\Omega}$ and $y + \alpha^{-1}e \in \Omega$. Thus the linear transformation $L(xy + x + y + \alpha^{-1}e)$ is positive definite due to Proposition 2.3, so is invertible. Therefore $xy = 0$. □

Remark 3.5 Theorem 3.4 is a generalization of Proposition 2.5 in [4] regarding second-order cones.

Theorem 3.6 *Let $\lambda \in (0, 1)$. The penalized Fischer–Burmeister function ψ_λ defined by $\psi_\lambda(x, y) = \lambda(x + y - (x^2 + y^2)^{1/2}) + (1 - \lambda)x_+y_+$ is a C-function.*

Proof Suppose that (x, y) satisfies (CC). Then $x_+y_+ = xy = 0$ and hence $\psi_\lambda(x, y) = \lambda(x + y - (x^2 + y^2)^{1/2}) = 0$ from Proposition 3.1 (iv).
Assume that $\psi_\lambda(x, y) = \lambda(x + y - (x^2 + y^2)^{1/2}) + (1 - \lambda)x_+y_+ = 0$. Multiplying by x_- both sides, we have

$$x_-\{\lambda(x + y - (x^2 + y^2)^{1/2}) + (1 - \lambda)x_+y_+\} = 0$$

or from $x_+x_- = 0$

$$\lambda\{-(x_-)^2 - x_-[(x^2 + y^2)^{1/2} - y]\} + (1 - \lambda)x_-(x_+y_+) = 0.$$

Applying the trace operator to both sides yields that

$$-\lambda\{\mathrm{tr}((x_-)^2) + \mathrm{tr}(x_-[(x^2 + y^2)^{1/2} - y])\} + (1 - \lambda)\mathrm{tr}(x_-(x_+y_+)) = 0.$$

Since both $(x^2 + y^2)^{1/2} - y$ (by the Löwner-Heinz inequality) and x_- belong to $\overline{\Omega}$, and the trace operator is associative, we see that

$$\mathrm{tr}((x_-)^2) = 0,$$

which implies that $x_- = 0$, hence $x \geq 0$. Similarly we obtain $y \geq 0$. Thus $\lambda(x + y - (x^2 + y^2)^{1/2}) + (1-\lambda)xy = 0$. So we have

$$(x^2 + y^2)^{1/2} = x + y + \alpha xy$$

where $\alpha = (1-\lambda)/\lambda$. Squaring both sides yields that

$$(2e + \alpha[(\alpha xy + x + y) + x + y])(xy) = 0$$

or

$$L(2e + \alpha[(\alpha xy + x + y) + x + y])(xy) = 0.$$

Since $2e + \alpha[(\alpha xy + x + y) + x + y] \in \Omega$, we conclude that $xy = 0$ as desired. □

Remark 3.7 Theorem 3.6 extends Proposition 1 of [3] regarding NCP case to symmetric cones. The proof above is motivated by an elegant proof given by Xin Chen in an unpublished note dealing with a SDCP version of Proposition 1 in [3].

4 A trace inequality and boundedness of level sets

In this section, we investigate the boundedness of level sets of the natural merit function induced by the penalized natural residual function $\phi_p(x, y) = x - [x - y]_+ + x_+ y_+$ by way of a new trace inequality in Euclidean Jordan algebras. Our approach shows that the boundedness of level sets of merit functions of SCCP is closely related to trace inequalities. First recall that a function $F : V \to V$ is said to be *monotone* if

$$\langle x - y,\ F(x) - F(y)\rangle \geq 0 \quad \forall x, \quad y \in V.$$

We begin with the following geometric lemma.

Lemma 4.1 *Let $\{x_n\}$ be a sequence in $\overline{\Omega}$ and $x \in \Omega$. If $\|x_n\|$ goes to infinity as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ such that $\langle x_{n_k},\ x\rangle \to \infty$* as *$k \to \infty$.*

Proof Consider the normalized sequence $y_n = x_n/\|x_n\|$. Then there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and a nonzero element $y \in \overline{\Omega}$ such that $y_{n_k} \to y$. Clearly $\langle y_{n_k},\ x\rangle \to \langle y,\ x\rangle$, a positive number. Indeed, as Ω is a symmetric cone, its open dual cone defined by

$$\Omega^* = \left\{x \in V \mid \langle y,\ x\rangle > 0, \quad \forall y \in \overline{\Omega}\backslash\{0\}\right\}$$

coincides with Ω ([6], p. 4), hence $\langle y,\ x\rangle > 0$. This implies that the sequence $\langle x_{n_k},\ x\rangle \to \infty$ because $\|x_{n_k}\|$ goes to infinity as $k \to \infty$. □

Using Lemma 4.1, we extend Lemma 3.1 of [4] to a symmetric cone case.

Lemma 4.2 *Let $F : V \to V$ be a monotone function. Suppose that Problem Eq. (3.4) has a strictly feasible point $\hat{x} \in \Omega$ such that $F(\hat{x}) \in \Omega$. For any sequence $\{x^k\}$ satisfying $\|x^k\| \to \infty$,* $\limsup_{k\to\infty} \|x_-^k\| < \infty$ *and* $\limsup_{k\to\infty} \|F(x^k)_-\| < \infty$*, we have a subsequence $\{x^{k_n}\}$ of $\{x^k\}$ such that*

$$\langle x_+^{k_n},\ F(x^{k_n})_+\rangle \to \infty.$$

Proof As F is monotone, we have

$$\langle x^k,\ F(\hat{x})\rangle + \langle \hat{x},\ F(x^k)\rangle \le \langle x^k,\ F(x^k)\rangle + \langle \hat{x},\ F(\hat{x})\rangle.$$

Decomposing $x^k = x^k_+ - x^k_-$ and $F(x^k) = F(x^k)_+ - F(x^k)_-$ yields that

$$\langle x^k_+,\ F(\hat{x})\rangle - \langle x^k_-,\ F(\hat{x})\rangle + \langle \hat{x},\ F(x^k)_+\rangle - \langle \hat{x},\ F(x^k)_-\rangle \le \langle x^k,\ F(x^k)\rangle + \langle \hat{x},\ F(\hat{x})\rangle.$$

By Lemma 4.1 and the assumption on $\{x^k\}$ there exists a subsequence $\{x^{k_n}\}$ of $\{x^k\}$ such that

$$\langle x^{k_n}_+,\ F(\hat{x})\rangle \to \infty.$$

It is easily checked that $\limsup_{k\to\infty}\langle x^k_-,\ F(\hat{x})\rangle < \infty,\ \limsup_{k\to\infty}\langle \hat{x},\ F(x^k)_-\rangle < \infty$ and $\langle \hat{x},\ F(x^k)_+\rangle \ge 0$. Thus we have

$$\langle x^{k_n},\ F(x^{k_n})\rangle \to \infty.$$

Since we can expand $\langle x^k,\ F(x^k)\rangle$ as

$$\langle x^k,\ F(x^k)\rangle = \langle x^k_+,\ F(x^k)_+\rangle + \langle x^k_-,\ F(x^k)_-\rangle - \langle x^k_+,\ F(x^k)_-\rangle - \langle x^k_-,\ F(x^k)_+\rangle,$$

we conclude that $\langle x^{k_n}_+,\ F(x^{k_n})_+\rangle \to \infty$. This completes the proof. □

Now we are in a position to introduce a trace inequality in Euclidean Jordan algebras for the purpose of dealing with the boundedness of level sets of the natural merit function induced by the penalized natural residual function $\phi_p(x, y) = x - [x - y]_+ + x_+y_+$. About thirth years ago, Thompson [17] obtained the matrix triangle inequality that for any two complex matrices A and B there exist unitary matrices U and V such that

$$|A + B| \le U|A|U^* + V|B|V^*$$

where $|A|$ denotes the unique square root of the positive semidefinite AA^*. Recently Thompson's inequality has been extended to matrix algebras over quaternions due to Farenick and Psarrakos [10, Theorem 2.3]. It induces in particular a trace inequality

$$\mathrm{tr}|A + B| \le \mathrm{tr}|A| + \mathrm{tr}|B|, \quad \forall A, B \in \mathrm{M}_n(\mathbb{C}).$$

We are interested in extending the trace inequality to Euclidean Jordan algebras that will play a key role to obtain the boundedness of the solution set of a monotone SCCP:

$$\mathrm{tr}|a + b| \le \mathrm{tr}|a| + \mathrm{tr}|b|, \quad \forall a, b \in V. \tag{4.8}$$

Proposition 4.3 *The trace inequality Eq. (4.8) holds true for any Euclidean Jordan algebras.*

Proof Let V be a Euclidean Jordan algebra with rank r. Let $\lambda : V \to \mathbb{R}^r$ be the eigenvalue function on V, $\lambda(x) = (\lambda_1(x), \ldots, \lambda_r(x))$, where $\lambda_1(x) \ge \cdots \ge \lambda_r(x)$ denote the eigenvalues of x in nonincreasing order. Let $\psi : \mathbb{R}^r \to [0, \infty)$ be the 1-norm on $\mathbb{R}^r$, that is, $\psi(x_1, x_2, \ldots, x_r) = \sum_{i=1}^r |x_i|$. Then ψ is a convex function and

$$\mathrm{tr}|a| = (\psi \circ \lambda)(a), \quad a \in V.$$

Applying the result of Baes [1, Theorem 6.3], the function $a \mapsto \mathrm{tr}|a| = (\psi \circ \lambda)(a)$ is convex and therefore $\mathrm{tr}|\frac{a+b}{2}| \le \frac{1}{2}(\mathrm{tr}|a| + \mathrm{tr}|b|)$ or $\mathrm{tr}|a + b| \le \mathrm{tr}|a| + \mathrm{tr}|b|$ for all $a, b \in V$ from the homogeneous property of the trace functional. □

Remark 4.4 An alternate proof of the trace inequality on simple Euclidean Jordan algebras can be obtained by the following variational expression. Let $B_\infty = \{a \in V : -e \preceq a \preceq e\}$, the set of all elements of V with eigenvalues in $[-1, 1]$. Then

$$\mathrm{tr}|a| = \max\{|\langle a, b\rangle| : b \in B_\infty\}.$$

We are indebted to Leonid Faybusovich for suggesting the following proof.

Proof Let $b \in B_\infty$ and let $k(b) = \sum_{i=1}^r \lambda_i(b)c_i$ be its spectral decomposition for some $k \in \mathrm{Aut}(V)$ consisting of all the Jordan automorphisms of V where a Jordan automorphism of V is defined to be an invertible linear transformation k of V such that $k(xy) = k(x)k(y)$. Consider the the Peirce decomposition of V with respect to the Jordan frame $\{c_i\}$: for $x \in V$, $x = \sum_{i=1}^r d_i(x)c_i + \sum_{i<j} x_{ij}$ (see [6, Theorem IV.2.1 (i)]). Let $\mathrm{diag}(x) = (d_1(x), \ldots, d_r(x))$ be the Horn-Schur map. Then

$$\begin{aligned}\big|\langle a, b\rangle\big| = \big|\langle k(a), k(b)\rangle\big| = \Big|\sum_{i=1}^r d_i(a)\lambda_i(b)\Big| &\le \max_{i\in[1,r]} \big|\lambda_i(b)\big| \sum_{i=1}^r \big|d_i(k(a))\big| \\ &\le \sum_{i=1}^r \big|d_i(k(a))\big|\end{aligned}$$

since $b \in B_\infty$ implies that $|\lambda_i(b)| \le 1$ for all i. Now, by the Horn-Schur convexity theorem due to Lim et al. [14, Theorem 3], the vector $(d_1(a), \ldots, d_r(a))$ is contained in the convex hull of the set $\{(\lambda_{\sigma(1)}(a), \ldots, \lambda_{\sigma(r)}(a)) : \sigma \in S^r\}$. Hence $||\mathrm{diag}(a)||_1 \le ||\lambda(a)||_1$ and therefore $|\langle a, b\rangle| \le ||\lambda(a)||_1 = \mathrm{tr}|a|, \quad \forall b \in B_\infty$.

If $a = \sum_{i=1}^r \lambda_i(a)c_i'$ is a spectral decomposition of a, take $b = \sum_{i=1}^r \mathrm{sign}(\lambda_i(a))c_i'$. Then $b \in B_\infty$ and $|\langle a, b\rangle| = ||\lambda(a)||_1$ which proves the claim. Now from the claim we have

$$\begin{aligned}\mathrm{tr}|a + c| &= \max\{|\langle a + c, b\rangle| : b \in B_\infty\} \le \max\{|\langle a, b\rangle| + |\langle c, b\rangle| : b \in B_\infty\} \\ &\le \max\{|\langle a, b\rangle| : b \in B_\infty\} + \max\{|\langle c, b\rangle| : b \in B_\infty\} = \mathrm{tr}|a| + \mathrm{tr}|c|.\end{aligned}$$

□

Remark 4.5 The trace inequality Eq. (4.8) is equivalent to

$$\mathrm{tr}|a - b| \le \mathrm{tr}(a) + \mathrm{tr}(b), \quad \forall a, b \succeq 0. \tag{4.9}$$

Indeed, Eq. (4.9) implies that

$$\begin{aligned}\mathrm{tr}|a + b| &= \mathrm{tr}|(a_+ + b_+) - (a_- + b_-)| \le \mathrm{tr}(a_+ + b_+) + \mathrm{tr}(a_- + b_-) \\ &= \mathrm{tr}(a_+ + a_-) + \mathrm{tr}(b_+ + b_-) = \mathrm{tr}|a| + \mathrm{tr}|b|.\end{aligned}$$

Proposition 4.6 *Let* $x, y \in V$. *If* $\max\{||x_-||,\ ||y_-||\} < C_0$, *then*

$$\mathrm{tr}(x - [x - y]_+) > -2\,\mathrm{rank}(V)C_0.$$

Proof Let $r = \mathrm{rank}(V)$. Then $\mathrm{tr}(x_-) \le r||x_-|| \le rC_0, \quad \mathrm{tr}(y_-) \le rC_0$.
From $x - [x - y]_+ = \frac{1}{2}(x + y - |x - y|)$, we have

$$\begin{aligned}2\mathrm{tr}(x - [x - y]_+) &= \mathrm{tr}(x + y - |x - y|) = \mathrm{tr}(x_+ - x_- + y_+ - y_- - |x - y|) \\ &= \mathrm{tr}(x_+) + \mathrm{tr}(y_+) - \mathrm{tr}(x_-) - \mathrm{tr}(y_-) - \mathrm{tr}(|x - y|) \\ &\ge \mathrm{tr}(x_+) + \mathrm{tr}(y_+) - \mathrm{tr}(x_-) - \mathrm{tr}(y_-) - \mathrm{tr}(|x|) - \mathrm{tr}(|y|)\end{aligned}$$

$$= -2\mathrm{tr}(x_-) - 2\mathrm{tr}(y_-)$$
$$> -4rC_0$$

where the first inequality follows from the trace inequality Eq. (4.9). □

Finally we reach the goal, the boundedness of level sets of the natural merit function induced by the penalized natural residual function under monotonicity and strict feasibility.

Theorem 4.7 *Let $C \geq 0$ is a constant. Then the level set $M = \{x \in V \mid \|\Phi_p(x)\| \leq C\}$, where $\Phi_p(x) = \phi_p(x, F(x))$, is bounded, provided that F is monotone and Problem Eq.* (3.4) *is strictly feasible, i.e., there exists a point $\hat{x} \in \Omega$ with $F(\hat{x}) \in \Omega$,*

Proof It is sufficient to prove that there is a subsequence $\{x^{k_n}\}$ of $\{x^k\}$ such that $\|\Phi_p(x^{k_n})\| \to \infty$ as $\|x^k\| \to \infty$. If $||x^k_-|| \to \infty$ or $||F(x^k)_-|| \to \infty$, the result holds by Proposition 3.3. Suppose that $\limsup_{k\to\infty} ||x^k_-|| < \infty$ and $\limsup_{k\to\infty} ||F(x^k)_-|| < \infty$. Then there exists a constant C_0 such that

$$\max\{||x^k_-||,\ ||F(x^k)_-||\} < C_0.$$

By Lemma 4.2 (using the strict feasibility), there is a subsequence $\{x^{k_n}\}$ of $\{x^k\}$ such that

$$\mathrm{tr}(x^{k_n}_+ F(x^{k_n})_+) = \langle x^{k_n}_+,\ F(x^{k_n})_+\rangle \to \infty.$$

Then we have

$$\begin{aligned}||x^{k_n} - [x^{k_n} - F(x^{k_n})]_+ + x^{k_n}_+ F(x^{k_n})_+|| &\geq \frac{1}{r}\mathrm{tr}(x^{k_n} - [x^{k_n} - F(x^{k_n})]_+ + x^{k_n}_+ F(x^{k_n})_+) \\ &= \frac{1}{r}\mathrm{tr}(x^{k_n} - [x^{k_n} - F(x^{k_n})]_+) \\ &\quad + \frac{1}{r}\mathrm{tr}(x^{k_n}_+ F(x^{k_n})_+)\end{aligned}$$

and by Proposition 4.6 we get

$$\liminf_{k\to\infty} \mathrm{tr}(x^{k_n} - [x^{k_n} - F(x^{k_n})]_+) > -\infty.$$

Therefore we conclude $||x^{k_n} - [x^{k_n} - F(x^{k_n})]_+ + x^{k_n}_+ F(x^{k_n})_+|| \to \infty$. □

Corollary 4.8 *If F is monotone and Problem Eq.* (3.4) *has a strictly feasible point, then the solution set of Problem Eq.* (3.4) *is bounded.*

Remark 4.9 Theorem 4.7 and Corollary 4.8 extend the corresponding Theorem 3.2 and Corollary 3.3 of [4] concerning SOCCP to symmetric cones including the case of SDCP.

5 Concluding remarks

The results of the previous sections in a sense only begin the theoretical study of C-functions and their merit functions for SCCP. They are mainly focused on the penalized versions of the two popular C-functions. As far as the boundedness of level sets of the natural merit functions induced from the penalized C-functions is concerned, only the case of the merit function induced by the penalized natural residual function was treated. So the next logical step in future research is to apply the new trace inequality method to prove the boundedness of level sets of the natural merit function induced from the penalized Fischer–Bermeister function in Euclidean Jordan algebras. As pointed out in [4], the Jacobian of $\Phi_p(x)$ may have lack of nonsingularity when $\Phi_p(x)$ is the natural merit function appeared in Theorem 4.7. In comparison to this, in the case of NCP, the penalized Fischer–Bermeister function has its rich and fruitful smoothness advantages over other NCP-functions as claimed in Chen et al. [3]. So there is an enough motivation to develop the corresponding extensions to symmetric cones.

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